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ON THE DE RHAM COHOMOLOGY OF THE LEAF SPACE OF A FOLIATION

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§1. INTRODUCTION

LET M be a C^∞ manifold and τ a smooth foliation of M, i.e. an involutive smooth sub-bundle of the tangent bundle $T(M)$ of M. Let $E_\tau^*(M)$ denote the space of smooth differential forms α on M such that

$$i_X\alpha = 0 = i_X d\alpha, \qquad X \in \Gamma(\tau) \tag{1.1}$$

where $\Gamma(\tau)$ is the space of smooth sections of τ. Equivalently, $\alpha \in E_\tau^*(M)$ if and only if

$$i_X\alpha = 0 = L_X\alpha, \qquad X \in \Gamma(\tau). \tag{1.2}$$

$E_\tau^*(M)$ is a subalgebra of $E^*(M)$, the algebra of differential forms on M. Exterior differentiation leaves $E_\tau^*(M)$ invariant, and we denote by $H_\tau^*(M)$ the cohomology algebra of the complex $(E_\tau^*(M), d)$.

$E_\tau^*(M)$ is, in some sense, the algebra of "smooth differential forms" on the leaf space M/τ, and $H_\tau^*(M)$ is the "de Rham cohomology" of M/τ, where we recall that M/τ is the topological space obtained by identifying each leaf of τ to a point. For example, suppose $\pi\colon M \to N$ is a differentiable fiber bundle whose fibers are the leaves of τ. Then M/τ is homeomorphic to N, $E_\tau^*(M) = \pi^*E^*(N)$, and $H_\tau^*(M)$ is canonically isomorphic to $H^*_{\text{de R}}(N)$, the de Rham cohomology algebra of N. If M is compact, then N is compact and $H_\tau^*(M)$ is finite dimensional.

Bott has asked whether M compact implies $H_\tau^*(M)$ is finite dimensional for every τ. Note that, in general, $H_\tau^0(M)$ is isomorphic to $H^0_{\text{de R}}(M)$ and that $H_\tau^1(M)$ injects into $H^1_{\text{de R}}(M)$, for if $\alpha \in E_\tau^1(M)$ and $\alpha = df$, then $f \in E_\tau^0(M)$. Hence M compact implies $H_\tau^0(M)$ and $H_\tau^1(M)$ are finite dimensional. However, it turns out that $H_\tau^i(M)$ can be of infinite dimension for $i \geq 2$. Below we construct examples of line fields τ_n on compact $(n+3)$-dimensional manifolds M_n such that

$$\dim H_{\tau_n}^i(M_n) = \infty, \qquad 2 \leq i \leq n+2. \tag{1.3}$$

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§2. THE EXAMPLES

Assume we have constructed τ_0 on M_0 satisfying (1.3). Let N be a compact C^∞ manifold, and let τ' denote the induced line field

$$0 \times \tau_0 \subseteq T(N) \times T(M_0) \simeq T(N \times M_0).$$

It is shown in [2] that the inclusion of double complexes

$$E^*(N) \otimes E\tau_0{}^*(M_0) \xrightarrow{i} E_{\tau'}{}^*(N \times M_0) \tag{2.1}$$

induces an isomorphism when cohomology is taken with respect to the first differential (the one induced by exterior differentiation on N). Hence i induces an isomorphism on cohomology and the Künneth formula shows

$$H_{\tau'}{}^*(N \times M_0) \simeq H^*_{\mathrm{de\,R}}(N) \otimes H\tau_0{}^*(M_0). \tag{2.2}$$

Let T^n denote the n-torus, let $M_n = T^n \times M_0$, and let τ_n be the induced line field on M_n. Then $(1.3)_n$ follows from (2.2). Alternately, the methods of [1] show $H_{\tau_n}{}^*(M_n)$ can be calculated from the T^n-invariant forms in $E_{\tau_n}{}^*(M_n)$, thus giving another proof of (2.2) for $N = T^n$.

We now construct M_0 and τ_0. Let $X = \dfrac{\partial}{\partial\theta}$ denote the standard vector field on S^1. If Y is a vector field on the complex plane $\mathbb{C}$, we let $X + Y$ denote the vector field on $\mathbb{C} \times S^1$ which is X in the S^1 direction and Y in the $\mathbb{C}$ direction, and we also let $X + Y$ denote the associated line field. Let D denote the unit disk in $\mathbb{C}$, and choose points $z_i \in \{\frac{1}{4} < |z| < 1\}$ whose set of accumulation points is ∂D. Choose smooth vector fields Y_i with disjoint support in $\{\frac{1}{4} \leq |z| \leq 1\}$ such that $\exp(tY_i)$ is rotation by an angle $\beta_i t$ in a neighborhood of z_i, where $0 < \beta_i < 1$. We may choose real numbers s_i, $0 < s_i < 1$, so that $s_1 Y_1 + s_2 Y_2 + \cdots$ converges in the C^∞ topology to a vector field Y. Then

$$\operatorname{supp} Y \subseteq \{\tfrac{1}{4} \leq |z| \leq 1\} \tag{2.3}$$

and

$$\exp(tY) \text{ is rotation by an angle } ts_i\beta_i \text{ in a neighborhood of } z_i, \qquad 0 < s_i\beta_i < 1. \tag{2.4}$$

LEMMA. *Let* $0 < s < 1$. *Then*

$$\alpha \in E^1_{X+sY}(\mathbb{C} \times S^1) \Rightarrow \int_{D \times \{\theta\}} d\alpha = 0, \qquad \theta \in S^1. \tag{2.5}$$

$$\exists \gamma \in E^2_{X+sY}(\mathbb{C} \times S^1) \quad \textit{with} \quad \int_{D \times \{\theta\}} \gamma \neq 0, \qquad \theta \in S^1, \tag{2.6}$$

and $\operatorname{supp} \gamma \subseteq D \times S^1$.

Proof. Let $\alpha \in E^1_{X+sY}(\mathbb{C} \times S^1)$. By Stokes' theorem, (2.5) follows if we can show α is zero on $\partial D \times \{\theta\}$, and α is zero on $\partial D \times \{\theta\}$ if $\alpha(z_i, \theta) = 0$ for all i. Since $Y(z_i) = 0$, (1.1) says that $\alpha(z_i, \theta)$ is a linear combination of dx and dy. Now $\exp(2\pi(X + sY))(z_i, \theta) = (z_i, \theta)$ and

$$\alpha(z_i, \theta) = \exp(2\pi(X + sY))^*\alpha(z_i, \theta) = R_{2\pi s s_i \beta_i}\alpha(z_i, \theta)$$

where $R_{2\pi s s_i \beta_i}$ is rotation by an angle of $-2\pi s s_i \beta_i$ in the (dx, dy)-plane, $0 < 2\pi s s_i \beta_i < 2\pi$. Hence $\alpha(z_i, \theta) = 0$ proving (2.5). Setting $\gamma(z, \theta) = f(z)\, dx \wedge dy$ where

$$\operatorname{supp} f \subseteq \{|z| \leq \tfrac{1}{4}\} \quad \text{and} \quad \int_D f(z)\, dx \wedge dy \neq 0$$

gives (2.6). *Q.E.D.*

Now let $D_1, D_2, \ldots$ be a sequence of disjoint closed disks in D, and (using translation and a change of scale) let $Y_1, Y_2, \ldots$ be "copies" of Y in $D_1, D_2, \ldots$. Choose reals s_i, $0 < s_i < 1$, so that $s_1 Y_1 + s_2 Y_2 + \cdots$ converges to a smooth vector field $\bar{Y}$ on $\mathbb{C}$. Considering $\mathbb{C}$ as S^2 minus a point, we see that $X + \bar{Y}$ extends to a smooth line field τ_0 on $M_0 = S^2 \times S^1$. By (2.5) and (2.6) and the construction of $\bar{Y}$, there are closed forms $\gamma_i \in E_{\tau_0}{}^2(M_0)$ with support in $D_i \times S^1$ giving rise to linearly independent elements of $H_{\tau_0}{}^2(M_0)$. We have proved $(1.3)_0$.

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